LARGE TIME BEHAVIOR IN WASSERSTEIN SPACES AND RELATIVE ENTROPY FOR BIPOLAR DRIFT-DIFFUSION-POISSON MODELS

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ABSTRACT. We shall prove asymptotic stability results for nonlinear bipolar drift-diffusion-Poisson Systems arising in semiconductor device modeling and plasma physics in one space dimension. In particular, we shall prove that, under certain structural assumptions on the external potential and on the doping profile, all solutions match for large times with respect to all q–Wasserstein distances. We also prove exponential convergence to stationary solutions in relative entropy via the so called entropy dissipation (or Bakry–Émery) method.

1. INTRODUCTION.

In this paper we shall study the Cauchy problem for the one–dimensional nonlinear bipolar drift-diffusion-Poisson model

$$\begin{cases} n_t = \left(f(n)_x + n\left[V_n(x) - \psi(t,x)\right]_x\right)_x \\ p_t = \left(f(p)_x + p\left[V_p(x) + \psi(t,x)\right]_x\right)_x \\ \psi_{xx} = n - p - C, \end{cases} \tag{1}$$

which arises in semiconductor device modeling and in plasma physics (see e. g. [MRS90, Jün01]). The initial data

$$n(t=0,x) = n_0(x), \qquad p(t=0,x) = p_0(x)$$

are both chosen in $L^1_+(\mathbb{R})$ and satisfy

$$\int_{-\infty}^{+\infty} n_0(x)dx = N_I, \qquad \int_{-\infty}^{+\infty} p_0(x)dx = P_I$$

for fixed nonnegative N_I, P_I. We shall consider nonlinearities f of the form

$$f(z) = z^m, \qquad m \geq 1.$$

In system (1), $n(t,x)$ denotes the spatial distribution of the (negatively charged) electrons, $p(t,x)$ is the distribution of the (positively charged) holes, $\psi(t,x)$ is the self-consistent electrostatic potential created by the two charge carriers (the electrons and the holes) and the doping profile $C(x)$ of the semiconductor material. As usual in this framework, C is supposed to belong to $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. In this model we additionally require that the charge carriers are confined by two (external) potentials V_n, V_p. Throughout this paper we shall work under the following structural assumptions on the external potentials and on the doping profile:

$$-\|C\|_{L^\infty(\mathbb{R})} + \min\left\{\inf_{\mathbb{R}} V_n'', \inf_{\mathbb{R}} V_p''\right\} =: \Lambda > 0. \tag{2}$$

Even though the condition (2) does not have a deep physical interpretation, it will appear very natural both in the result involving the decay of Wasserstein distances and in the computation of the relative entropy.

As is well–known, the divergence form of (1) implies $\int_{-\infty}^{+\infty} n(x,t)dx = N_I$ and $\int_{-\infty}^{+\infty} p(x,t)dx = P_I$ for all times $t > 0$. Moreover, a minimum principle ensures that n and p remain nonnegative for positive times (later on we shall also need a stronger version of the minimum principle, which is proven in Appendix A).

The main result of our paper concerns with stability properties the solutions to the model (1) in context of the space of probability measures $\mathcal{P}(\mathbb{R})$. More precisely, the function $t \mapsto (n(t), p(t))$ is interpreted (after mass normalization) as a curve in the product space $\mathcal{P}(\mathbb{R}) \times \mathcal{P}(\mathbb{R})$ endowed with all the q–Wasserstein distances with $q \geq 2$. Under certain natural assumptions on the initial data and under the structural condition (2), we prove that all the solution orbits match exponentially fast for large times, in a similar fashion as in the case of scalar nonlinear drift–diffusion models described in [Ott01, CMV03]. Our result is also valid for the ∞–Wasserstein distance, which makes sense only in the case of nonlinear diffusion, where finite speed of propagation of the support holds. Therefore, in such case we prove that the 'distance' between the supports of any two solutions tends to zero for large times. Our result uses the interpretation of Wasserstein distances in terms of integral norms involving pseudo–inverses, as described in [CT05]. Consequently, it only applies to the case of one space dimension. So far (to our knowledge), such an approach has been mostly used in the study of *scalar* drift–diffusion equations. We shall prove that the problem of coupling between the two transport equations in (1) can be solved easily by employing additional results involving L^1–decay to stationary solutions. We shall describe the aforementioned stability result concerning with the Wasserstein distances in Section 2.

The asymptotic behavior via entropy methods for the model (1) has been extensively studied in the recent years. We mention in particular [AMT00], where the convergence to stationary states in relative entropy has been proven in the case of linear diffusion and in several space variables. An analogous result has been proven in [BDM01] in the case of nonlinear diffusion. In both papers, the use of so called *Log–Sobolev type inequalities* is crucial in order to achieve the desired time decay (see also [AMTU01, CT00, CJM$^+$01]). In particular, the validity of a suitable inequality of that type is proven in case of linear diffusion in [AMT00] by a perturbation argument which requires an *a priori* L^∞–estimate for the Newtonian potential ψ. Such a situation occurs when the space dimension is strictly larger than 2. In the present paper, we shall apply the Bakry–Émery *entropy – entropy dissipation* strategy (see [BÉ85, Vil03]), which consists in computing the second derivative w.r.t. time of a suitable entropy functional. The main advantage of such a strategy is that the rate of convergence to stationary states does not depend on the size of the initial data. Moreover, we can cover the one–dimensional case where the L^∞–estimate of the Newtonian potential does not hold. The price we pay is that we need assumption (2) (which is not needed in the aforementioned papers) in order to achieve the exponential decay of the entropy dissipation. However, here we don't need V_n'' and V_p'' to be uniformly bounded as required in [AMT00]. We remark that (similarly to [BDM01]) the result in case of nonlinear diffusion is only formal, in the sense that one has to assume that the solution enjoys enough regularity in order to compute the evolution of the entropy functional. Our result concerning the entropy method is contained in Section 3.

2. Stability in Wasserstein spaces

Let $\mathcal{P}$ denote the space of probability measures on $\mathbb{R}$. In order to interpret the solution orbit of (1) as a curve in the product space $\mathcal{P}\times\mathcal{P}$, we set

$$\underline{n} := \frac{1}{N_I} n, \qquad \underline{p} := \frac{1}{P_I} p. \tag{3}$$

Now it makes sense to study the evolution of $\underline{n}(t)$ and $\underline{p}(t)$ in the context of the q–Wasserstein distances. We observe that $\underline{n}$ and $\underline{p}$ satisfy the same evolution equations as n and p in (1), whereas the elliptic equation in (1) can be reformulated as

$$\psi_{xx} = N_I \underline{n} - P_I \underline{p} - C.$$

Let us denote by $\mathcal{P}_q$ the set of Borel probability measures on $\mathbb{R}$ with finite q-th moment, $q \geq 1$. Then, the Wasserstein distance of order q on $\mathcal{P}_q \times \mathcal{P}_q$ is defined as

$$W_q(\mu,\nu) \equiv \inf_{\pi\in\Pi(\mu,\nu)} \left(\int_{-\infty}^{\infty} |x-y|^q d\pi(x,y) \right)^{1/q},$$

$\Pi(\mu,\nu)$ denoting the set of all probability measures on $\mathbb{R}^2$ with marginals μ,ν, respectively (cf. [Vil03]). Let $F(x) \equiv \mu(-\infty,x]$, $G(y) \equiv \nu(-\infty,y]$ be the respective cumulative distribution functions of the absolutely continuous probability measures $d\mu(x) = f(x)dx$ and $d\nu(y) = g(y)dy$. If the probability measure μ does not charge small sets (i.e., sets of zero Hausdorff measure), then $T = G^{-1}\circ F$ transports μ onto ν, in the sense that $\nu(B) = \mu(T^{-1}(B))$ for all Borel sets $B \subseteq \mathbb{R}$. This easily yields the following formula (see [Vil03])

$$W_q(\mu,\nu)^q = \int_0^1 |F^{-1}(\xi) - G^{-1}(\xi)|^q d\xi,$$

where

$$F^{-1}(\xi) \equiv \inf\{x\in\mathbb{R} : F(x) > \xi\}, \quad \xi\in[0,1]$$

denotes the pseudo–inverse of F. The above representation of the Wasserstein distance on the real line is very useful in the study of certain diffusion equations (see e.g. [CGT04, CT05]). In short, one simply has to compute the corresponding equation for the pseudo-inverses of the primitives of the solutions (after mass normalization), and measure their distance in $L^p(0,1)$ in order to achieve an estimate of the q–Wasserstein distance.

Let us then set

$$N(t,x) = \int_{-\infty}^{x} \underline{n}(t,y)dy, \qquad P(t,x) = \int_{-\infty}^{x} \underline{p}(t,y)dy.$$

Let N^{-1} and P^{-1} denote the pseudo–inverses of N and P respectively. We recall that N^{-1} and P^{-1} are defined on the set $\{(\xi,t),\ 0<\xi<1,\ t\geq 0\}$, whereas $N^{-1}(t)$ ($P^{-1}(t)$ respectively) can be continuously extended to the points $\xi = 0$ and $\xi = 1$ if and only if $n(t)$ ($p(t)$ respectively) has compact support. With this notations, we can formally deduce the partial differential equation satisfied by $N^{-1}(\xi,t)$ (we refer to [CT05] for further details)

$$N_t^{-1} = -\left((N_\xi^{-1})^{-m}\right)_\xi - V_n'(N^{-1}) + \psi_x|_{x=N^{-1}(t,\xi)}. \tag{4}$$

Observe that

$$\psi_x(t,x)|_{x=N^{-1}(t,\xi)} = N_I\xi - P_I P(t,N^{-1}(t,\xi)) - \mathcal{C}(N^{-1}(t,\xi)),$$

where we used the notation $\mathcal{C}(x) = \int_{-\infty}^x C(y)dy$. Thus, equation (4) becomes

$$N_t^{-1} = -\left((N_\xi^{-1})^{-m}\right)_\xi - V_n'(N^{-1}) + N_I\xi - P_I P(t,N^{-1}) - \mathcal{C}(N^{-1}). \tag{5}$$

Analogously, we get a similar equation for P^{-1}, namely

$$P_t^{-1} = -\left((P_\xi^{-1})^{-m}\right)_\xi - V_p'(P^{-1}) - N_I N(t,P^{-1}) + P_I\xi + \mathcal{C}(P^{-1}). \tag{6}$$

We are now ready to state our main theorem, in which we shall also refer to the entropy functional $\mathcal{E}$ defined later on in (11).

Theorem 2.1. *Let (n,p) and $(\tilde{n},\tilde{p})$ be two solutions to (1) with initial data n_0, p_0, $\tilde{n}_0$, $\tilde{p}_0$ belonging to $L^1_+(\mathbb{R})$ and such that $\mathcal{E}(n_0,p_0)$ and $\mathcal{E}(\tilde{n}_0,\tilde{p}_0)$ are finite. Let $(\underline{n}(t),\underline{p}(t))$ and $(\underline{\tilde{n}}(t),\underline{\tilde{p}}(t))$ be defined as in (3). Let $q \geq 2$ and let the integer k be defined by $k = k(q) := \left[\frac{q+2}{2}\right]$. If the moments of order $2k$ of the initial data are finite, then for any $t \geq 0$ we have*

$$\left[W_q(\underline{n}(t),\underline{\tilde{n}}(t)) + W_q(\underline{p}(t),\underline{\tilde{p}}(t))\right] \leq Ce^{-\Lambda t}, \tag{7}$$

where $C > 0$ is a constant depending only on the initial data and Λ is defined by the structural assumption (2).

Proof. We shall prove the assertion (7) in the case $q = 2k$ and k is a positive integer. The general result then comes by simple L^q interpolation. Let N, $\tilde{N}$, P, $\tilde{P}$ be the distribution functions of $\underline{n}$, $\underline{\tilde{n}}$, $\underline{p}$ and $\underline{\tilde{p}}$ respectively. In order to prove statement (7), we shall perform a direct L^{2k} formal computation on the pseudo–inverse equations (5) and (6). As usual in this framework (see [CGT04]), the computation can be made rigorous by approximating the solutions to the Cauchy problem for (1) by solutions of the IBV problem for (1) on a bounded interval with uniformly positive initial data and zero flux boundary conditions. In order to employ such an approximation argument one needs the validity of a minimum principle which guarantees that the solution of the approximating IBV problem stays uniformly positive for all $t > 0$ if so is the initial datum. We shall prove such a property in Appendix A. Moreover, classical energy estimates on the approximating problem in the spirit of [AMT00, Section 2] provide enough compactness for the approximating sequence in order to have consistency in the limit, where one also employs weak lower semi–continuity properties of the Wasserstein distances (see [Vil03]). We refer to [CGT04] for further details about this approximating procedure.

As a consequence of the above considerations, the pseudo–inverses N^{-1} and P^{-1} can be considered as *real* inverses (because N and P are strictly monotone). Moreover, they enjoy enough regularity in order to perform integration by parts. The boundary term eventually appearing after integration by parts can be dropped out due to the finiteness of the domain in the approximating argument (see [CGT04]). Therefore, we use (5) to compute

$$
\begin{aligned}
&\frac{d}{dt}W_{2k}(\underline{n}(t,.),\underline{\tilde{n}}(t,.))^{2k}\frac{d}{dt}\left\|N^{-1}(t,.)-\tilde{N}^{-1}(t,.)\right\|_{L^{2k}(0,1)}^{2k}\\
&=2k\int_0^1\left[N^{-1}(t,\xi)-\tilde{N}^{-1}(t,\xi)\right]^{2k-1}\left[-\frac{\partial}{\partial\xi}\left(\frac{1}{N_\xi^{-1}(t,\xi)^m}-\frac{1}{\tilde{N}_\xi^{-1}(t,\xi)^m}\right)\right.\\
&-\left[V_n'\left(N^{-1}(t,\xi)\right)-V_n'\left(\tilde{N}^{-1}(t,\xi)\right)\right]-P_I\left[P(t,N^{-1}(t,\xi))-\tilde{P}(t,\tilde{N}^{-1}(t,\xi))\right]\\
&\left.-\left[\mathcal{C}\left(N^{-1}(t,\xi)\right)-\mathcal{C}\left(\tilde{N}^{-1}(t,\xi)\right)\right]\right]d\xi. \qquad (8)
\end{aligned}
$$

After integration by parts, the first addend of the RHS in (8) becomes

$$
2k(2k-1)\int_0^1\left[N^{-1}-\tilde{N}^{-1}\right]^{2k-2}[N_\xi^{-1}-\tilde{N}_\xi^{-1}]\left(\frac{1}{(N_\xi^{-1})^m}-\frac{1}{(\tilde{N}_\xi^{-1})^m}\right)d\xi
$$

and the above expression is non positive due to the decreasing monotonicity of $u\mapsto u^{-m}$. The term involving the external potential V_n in (8) can be estimated as follows

$$
\begin{aligned}
&-2k\int_0^1\left[N^{-1}(t,\xi)-\tilde{N}^{-1}(t,\xi)\right]^{2k-1}\left[V_n'\left(N^{-1}(t,\xi)\right)-V_n'\left(\tilde{N}^{-1}(t,\xi)\right)\right]d\xi\\
&\leq-2k\left[\inf_{x\in\mathbb{R}}V_n''(x)\right]\left\|N^{-1}(t,.)-\tilde{N}^{-1}(t,.)\right\|_{L^{2k}(0,1)}^{2k}.
\end{aligned}
$$

A similar estimate holds for the term containing the doping profile C

$$
\begin{aligned}
&-2k\int_0^1\left[N^{-1}(t,\xi)-\tilde{N}^{-1}(t,\xi)\right]^{2k-1}\left(\mathcal{C}(N^{-1}(t,\xi))-\mathcal{C}(\tilde{N}^{-1}(t,\xi))\right)d\xi\\
&\leq 2k\|C\|_\infty\left\|N^{-1}(t,.)-\tilde{N}^{-1}(t,.)\right\|_{L^{2k}(0,1)}^{2k}.
\end{aligned}
$$

Let us now consider the term in (8) where the positive charge carrier appears,

$$
\begin{aligned}
&\quad -2kP_I\int_0^1\left(N^{-1}-\tilde{N}^{-1}\right)^{2k-1}\left(P\circ N^{-1}-\tilde{P}\circ\tilde{N}^{-1}\right)d\xi\\
&= -2kP_I\int_0^1\left(N^{-1}-\tilde{N}^{-1}\right)^{2k-1}\left(P\circ N^{-1}-P\circ\tilde{N}^{-1}\right)d\xi\\
&\quad -2kP_I\int_0^1\left(N^{-1}-\tilde{N}^{-1}\right)^{2k-1}\left(P\circ\tilde{N}^{-1}-\tilde{P}\circ\tilde{N}^{-1}\right)d\xi=:J_1+J_2
\end{aligned}
$$

Since P is nondecreasing, the term J_1 above is nonpositive. Moreover, by Hölder's inequality,

$$
\begin{aligned}
J_2 &\leq 2kP_I\left\|N^{-1}(t,.)-\tilde{N}^{-1}(t,.)\right\|_{2k}^{2k-1}\left[\int_0^1\left(P\circ\tilde{N}^{-1}-\tilde{P}\circ\tilde{N}^{-1}\right)^{2k}d\xi\right]^{1/2k}\\
&\leq 2kP_I\left\|N^{-1}(t,.)-\tilde{N}^{-1}(t,.)\right\|_{2k}^{2k-1}\|\underline{p}(t)-\underline{\tilde{p}}(t)\|_{L^1}.
\end{aligned}
$$

By combining all the above estimates, we obtain

$$
\begin{aligned}
\frac{d}{dt}W_{2k}(\underline{n}(t),\underline{\tilde{n}}(t))^{2k}\leq&-2k(\inf V_n''-\|C\|_\infty)W_{2k}(\underline{n}(t),\underline{\tilde{n}}(t))^{2k}\\
&+2kP_I\|\underline{p}(t)-\underline{\tilde{p}}(t)\|_{L^1}W_{2k}(n(t),\tilde{n}(t))^{2k-1}.
\end{aligned}
$$

Similarly, we can get the following estimate for the positive charge carriers,

$$\begin{aligned}\frac{d}{dt}W_{2k}(\underline{p}(t),\underline{\tilde{p}}(t))^{2k} \leq & -2k(\inf V_p'' - \|C\|_\infty)W_{2k}(\underline{p}(t),\underline{\tilde{p}}(t))^{2k} \\ & + 2kN_I\|\underline{n}(t)-\underline{\tilde{n}}(t)\|_{L^1}W_{2k}(\underline{p}(t),\underline{\tilde{p}}(t))^{2k-1}.\end{aligned}$$

In order to simplify the notation, let us set

$$\mathcal{X}_k(t) := W_{2k}(\underline{n}(t),\underline{\tilde{n}}(t))^{2k} + W_{2k}(\underline{p}(t),\underline{\tilde{p}}(t))^{2k}.$$

Due to assumption (2) and to the result in the Theorem 3.1, we can find a constant $C>0$ depending only on the initial data such that

$$\frac{d}{dt}\mathcal{X}_k(t) \leq -2\Lambda\mathcal{X}_k(t) + Ce^{-\frac{2\Lambda}{\alpha}t}\mathcal{X}_k(t)^{\frac{2k-1}{2k}} \leq -2k(2\Lambda - Ce^{-\frac{2\Lambda}{\alpha}t})\mathcal{X}_k(t) + 2Cke^{-\frac{2\Lambda}{\alpha}t},$$

where $\alpha=\max\{2,m\}$. Therefore, a first use of the variation of constants formula implies the estimate

$$\mathcal{X}_k(t) \leq Ce^{-\min(\frac{2\Lambda}{\alpha},2k)t},$$

which can be plugged into the above inequality in order to improve the rate of decay of $\mathcal{X}_k(t)$. After a finite number of iterations (depending on k), one gets the desired estimate (7). □

As already pointed out in the introduction, the result in the previous theorem can be extended to $k=\infty$ in case of nonlinear diffusions, due to the finite rate of growth of the support in that case. We recall that, given two compactly supported probability densities f,g, and their cumulative distribution functions F,G, we have

$$\lim_{p\to+\infty} W_p(f,g) = W_\infty(f,g) = \|F^{-1}-G^{-1}\|_{L^\infty([0,1])}$$

and therefore

$$\max\left(|\inf\operatorname{supp}(f) - \inf\operatorname{supp}(g)|, |\sup\operatorname{supp}(f) - \sup\operatorname{supp}(g)|\right) \leq W_\infty(f,g).$$

Consequently, we can easily send $k\to+\infty$ in (7) and prove the following result.

Corollary 2.2. *Let $f(z)=z^m$ with $m>1$. Let $(n_0,p_0),(\tilde{n}_0,\tilde{p}_0)$ be two compactly supported initial data for system (7). Then, the corresponding solutions $(n,p),(\tilde{n},\tilde{p})$ satisfy*

$$\begin{aligned}&|\inf\operatorname{supp}(n(t)) - \inf\operatorname{supp}(\tilde{n}(t))| + |\sup\operatorname{supp}(n(t)) - \sup\operatorname{supp}(\tilde{n}(t))| \\ &+ |\inf\operatorname{supp}(p(t)) - \inf\operatorname{supp}(\tilde{p}(t))| + |\sup\operatorname{supp}(p(t)) - \sup\operatorname{supp}(\tilde{p}(t))| \leq Ce^{-\Lambda t}, \quad (9)\end{aligned}$$

where C only depends on the initial data.

Remark 2.3. As a trivial consequence of the above result, one can prove the finite speed of propagation of the support of *any* solution to (1) in case $m>1$ by plugging the compactly supported stationary solution (n^∞,p^∞) defined later on in (10) into inequality (9).

3. Entropy dissipation method

The aim of this section is to prove exponential L^1- decay to stationary solutions via the entropy method for our model (1). This task has been already performed in more than 2 space dimensions in [AMT00, BDM01] by direct use of Log–Sobolev type inequalities at the level of the entropy identity. Our contribution to the theory is the computation of the time derivative of the entropy dissipation in the spirit of [BÉ85] (see also [AMTU01, CT00]). Our strategy does not require the use of the Holley–Stroock perturbation lemma, which does not apply to the one–dimensional case. We shall always work under the structural assumption (2). We recall that the enthalpy associated to the nonlinearity $f(z)$ is defined as

$$h(y) = \int_1^y \frac{f'(z)}{z}\, dz.$$

The stationary states $(n^\infty, p^\infty, \psi^\infty)$ of (1) are the solutions of

$$\begin{cases} n^\infty(x) = h^{-1}\left(C_n - V_n(x) + \psi^\infty(x)\right)_+ \\ p^\infty(x) = h^{-1}\left(C_p - V_p(x) + \psi^\infty(x)\right)_+ \\ \psi^\infty_{xx} = n^\infty - p^\infty - C, \end{cases} \tag{10}$$

where the constants C_n and C_p are uniquely determined in terms of the initial total masses N_I and P_I. For the existence results concerning the stationary states mentioned above see [DMU01] and the references therein. For a given solution $(n(t), p(t)$ to (1), we shall use the notation

$$v(t,x) = (n(t,x), p(t,x))$$
$$v^\infty(x) = (n^\infty(x), p^\infty(x))$$

for notational convenience of the next. We denote by $\mathcal{E}$ the entropy functional defined by

$$\mathcal{E}(v(t)) = \int (\Phi(n) + n(V_n - \psi^\infty))\, dx + \int (\Phi(p) + p\ (V_p + \psi^\infty))\, dx + \frac{1}{2}\int (\psi - \psi^\infty)_x^2 dx. \tag{11}$$

In this notation, $\Phi(x) = \int_0^x h(y)\, dy$. The relative entropy is

$$\mathcal{E}(v|v^\infty) = \mathcal{E}(v) - \mathcal{E}(v^\infty).$$

For further use we define

$$y(t,x) = (h(n) + V_n - \psi)_x\,, \quad \tilde{y}(t,x)\, (h(p) + V_p + \psi)_x\,,$$

such that the drift-diffusion equations in (1) become

$$n_t(t,x) = (ny)_x(t,x) \qquad p_t(t,x) = (p\tilde{y})_x(t,x).$$

We recall the following generalized *Csiszár–Kullback* inequality, which provides an upper bound of the L^1 norm of the difference between any positive density v and the ground state v^∞ in terms of their relative entropy. More precisely,

$$\|n - n^\infty\|^\alpha_{L^1(\mathbb{R})} + \|p - p^\infty\|^\alpha_{L^1(\mathbb{R})} \le C\mathcal{E}(v|v^\infty), \tag{12}$$

where $C > 0$ depends on N_I and P_I and $\alpha = \max\{2, m\}$.

Theorem 3.1. *With the notation introduced above and with Λ defined in (2), we have*

$$\mathcal{E}(v(t)|v^\infty) \le \mathcal{E}(v^0|v^\infty)\ \exp(-2\Lambda\, t).$$

The Csiszár-Kullback inequality (12) then implies

$$\|n(t,.) - n^\infty(.)\|_1 + \|p(t,.) - p^\infty(.)\|_1 \le C \exp\left(-\frac{2\Lambda}{\alpha}\, t\right),$$

where C only depends on the initial data.

Proof. As in [AMT00, BDM01], we have the entropy identity

$$\frac{d}{dt}\mathcal{E}(t) = -\int n\, y^2\, dx - \int p\, \tilde{y}^2 =: -\mathcal{I}(t).$$

We calculate the time derivative of $\mathcal{I}(t)$,

$$\begin{aligned}
\frac{d}{dt}\mathcal{I}(t) &= \int \left((ny)_x y^2 + (p\tilde{y})_x \tilde{y}^2\right)\, dx + 2\left[\int (ny\, y_t + p\tilde{y}\, \tilde{y}_t)\, dx\right] \\
&= -2\int \left(ny^2 y_x + p\tilde{y}^2\tilde{y}_x\right) dx + 2\int (ny(h(n)_{xt} - \psi_{xt}))\, dx + 2\int (p\tilde{y}(h(p)_{xt} + \psi_{xt}))\, dx \\
&= -2\int \left(ny^2 V_n'' + p\tilde{y}V_p''\right)\, dx - 2\int \left(ny^2(h(n)_{xx} - \psi_{xx}) + p\tilde{y}^2(h(p)_{xx} + \psi_{xx})\right) dx \\
&+ 2\int (ny\left((ny)_x h'(n)\right)_x + p\tilde{y}\left((p\tilde{y})_x h'(p)\right)_x)\, dx - 2\int (ny(ny - p\tilde{y}) - p\tilde{y}(ny - p\tilde{y}))\, dx,
\end{aligned}$$

where we have used

$$\psi_{xt} = \int_{-\infty}^{x} (n_t - p_t)\, dx = ny - p\tilde{y}.$$

Hence, we have

$$\begin{aligned}
\frac{d}{dt}\mathcal{I}(t) &\le -2\min\left(\inf V_n'', \inf V_p''\right)\, \mathcal{I}(t) - 2\int \left(ny^2 h(n)_{xx} - (ny)_x^2 h'(n)\right)\, dx \\
&- 2\int \left(p\tilde{y}h(p)_{xx} - (p\tilde{y})_x^2 h'(p)\right) dx + 2\int \left(ny^2\psi_{xx} - p\tilde{y}^2\psi_{xx}\right) dx - 2\int (ny - p\tilde{y})^2\, dx \\
&\le -2\min\left(\inf V_n'', \inf V_p''\right)\, \mathcal{I}(t) - 2\int \left(h'(n)(ny_x)^2 + h'(p)(p\tilde{y}_x)^2\right)\, dx \\
&- 2\int \left(ny^2 - p\tilde{y}^2)\right)(n-p) dx - 2\int (ny - p\tilde{y})^2\, dx + 2\|C\|_{L^\infty}\mathcal{I}(t). \qquad (13)
\end{aligned}$$

Since n and p are nonnegative, one can easily prove the following inequality

$$(ny^2 - p\tilde{y}^2)(n - p) \le (ny - p\tilde{y})^2. \qquad (14)$$

Inequality (14) and assumption (2) can be plugged into the above estimate (13) in order to obtain

$$\frac{d}{dt}\mathcal{I}(t) \le -2\Lambda\mathcal{I}(t). \qquad (15)$$

Starting from (13) one can apply the so called Bakry–Émery strategy (see also [AMTU01, CT00, Ott01, Vil03, MV00]) in order obtain the Log–Sobolev type inequality

$$\mathcal{E}(v^0|v^\infty) \le \frac{1}{2\Lambda}\mathcal{I}(v^0),$$

(which holds for arbitrary v_0 having finite entropy) and get the desired exponential decay for the relative entropy, thus completing the proof. Since this procedure is by now standard, we shall omit it. □

Remark 3.2. We stress here that the rate of decay obtained in Theorem 3.1 is independent of the size of the initial data. This fact constitutes an improvement of the results in [AMT00, BDM01] in case assumption (2) is satisfied. As already pointed out in the introduction, we remark that the above result holds as long as the solution enjoys enough regularity in order to give sense to the above calculations. Therefore, the result in Theorem 3.1 is only formal in the case of a nonlinear diffusion (see also [BDM01]).

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Appendix A.
A minimum principle

Theorem A.1 (Strong minimum principle)**.** *Let (n,p) solve (1) on a bounded interval with zero flux boundary conditions. Suppose (2) holds and suppose the initial datum satisfies $n(0) \geq k > 0$ and $p(0) \geq k > 0$. Then , $n(t) \geq k$ and $p(t) \geq k$, for all $t > 0$.*

Proof. Let $\eta_\varepsilon(z)$ be a smooth, nonnegative, nondecreasing, convex regularization of the positive part $(z)_+$, in particular such that $\eta_\varepsilon(z) \to (z)_+$ for any $z \in \mathbb{R}$ and for $\varepsilon \to 0$. We compute

$$\begin{aligned}
&\frac{d}{dt}\left[\int \eta_\varepsilon(k-n)dx + \int \eta_\varepsilon(k-p)dx\right] \\
&= -\int \eta_\varepsilon'(k-n)((f(n)_x + n(V_n-\psi)_x)_x dx - \int \eta_\varepsilon'(k-p)((f(p)_x + p(V_p+\psi)_x)_x dx \\
&= -\int \eta_\varepsilon''(k-n)f'(n)(n_x)^2 dx - \int \eta_\varepsilon''(k-p)f'(p)(p_x)^2 dx \\
&-\int (\eta_\varepsilon(k-n) + n\eta_\varepsilon'(k-n))(V_n'' - \psi_{xx})dx - \int (\eta_\varepsilon(k-p) + p\eta_\varepsilon'(k-p))(V_p'' + \psi_{xx})dx \\
&\leq (\|C\|_\infty - \inf V_n'')\int (\eta_\varepsilon(k-n) + n\eta_\varepsilon'(k-n))dx \\
&+ (\|C\|_\infty - \inf V_p'')\int (\eta_\varepsilon(k-p) + p\eta_\varepsilon'(k-p))dx + \int (g_\varepsilon(n) - g_\varepsilon(p))(n-p)dx,
\end{aligned}$$

where $g_\varepsilon(z) := \eta_\varepsilon(k-z) + n\eta_\varepsilon'(k-z)$. It is easily seen that g_ε is nonincreasing. Therefore, thanks to assumption (2) we obtain

$$\frac{d}{dt}\left[\int \eta_\varepsilon(k-n)dx + \int \eta_\varepsilon(k-p)dx\right] \leq 0,$$

which implies, in the limit as $\varepsilon \to 0$,

$$\left[\int (k - n(t))_+ dx + \int (k - p(t))_+ dx\right] \leq \left[\int (k - n_0)_+ dx + \int (k - p_0)_+ dx\right] = 0,$$

and the proof is complete. $\square$

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